Would You Like a Cold Beverage with Your Inquiry?

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Outline

1. *Cola Corporation*, an opportunity for inquiry (Interviewing M)
2. Student thinking
3. M's Question, extensions of the inquiry
4. Student thinking
5. Why use this activity

Cola Corporation: Launch

The *Cola Corporation* packages their soda in the following sizes:

- Individual cans
- Packs, consisting of 6 individual cans
- Cases, consisting of 6 packs
- Palettes, consisting of 6 cases
- Crates, consisting of 6 palettes
- Flotillas, consisting of 6 crates
- Other sizes are analogous but not named yet.

When making a shipment, the *Cola Corporation* ships the exact order using the largest possible (full) packages without going over the order.

Cola Corporation: Questions to Explore

(1) CSULB submits an order for $5,000$ cans. How will it be packed for shipping?

(2) CSULB calls the *Cola Corporation* to say that the previous order was an error and should be doubled. How will the new order be packed for shipping? Answer this question using at least two different strategies and check that they agree.

(3) LBCC has a standing order with the *Cola Corporation* that whenever an order of size S is shipped to CSULB, an order of size $S/3$ is shipped to LBCC. How will the order be packed for shipping?

Question 1

(1) CSULB submits an order for $5,000$ cans. How will it be packed for shipping?

Question 1: Strategy 1

Repeated/Block subtraction:

- $5000 - 3 * 6^4 = 1112 < 1296 = 6^4$, so 3 crates
- $1112 - 5 * 6^3 = 32 < 216 = 6^3$, so 5 palettes
- $32 - 0 * 6^2 = 32 < 36 = 6^2$, so 0 cases
- $32 - 5 * 6^1 = 2 < 6^1$, so 5 packs, and
- 2 cans

Question 1: Strategy 2

Integer part:

- $5000/6^4 \approx 3.86$, so 3 crates with 1112 cans left
- $1112/6^3 \approx 5.12$, so 5 palettes with 32 cans left
- $32/6^2 \approx 0.89$, so 0 cases
- $32/6^1 \approx 5.33$, so 5 packs, and
- 2 cans

Question 2

(2) CSULB calls the *Cola Corporation* to say that the previous order was an error and should be doubled. How will the new order be packed for shipping? Answer this question using at least two different strategies and check that they agree.

Question 2: Strategy 1

Convert $10,000$:

- $10000 - 1 * 7776 = 2224 < 7776$, so 1 flotilla
- $2224 - 1 * 1296 = 928 < 1296$, so 1 crate
- $928 - 4 * 216 = 64 < 216$, so 4 palettes
- $64 - 1 * 36 = 28 < 36$, so 1 case
- $28 - 4 * 6 = 4 < 6$, so 4 packs, and
- 4 cans

Question 2: Strategy 2

Double the packages:

$$2\left(\frac{2}{6^0},\frac{5}{6^1},\left(\frac{0}{6^2}\right),\frac{5}{6^3},\frac{3}{6^4}\right)$$

$$=\frac{4}{6^0},\frac{10}{6^1},\left(\frac{0}{6^2}\right),\frac{10}{6^3},\frac{6}{6^4}=\frac{4}{6^0},\frac{10}{6^1},\left(\frac{0}{6^2}\right),\frac{10}{6^3},\frac{0}{6^4},\frac{1}{6^5}$$

$$=\frac{4}{6^0},\frac{10}{6^1},\left(\frac{0}{6^2}\right),\frac{4}{6^3},\frac{1}{6^4},\frac{1}{6^5}=\frac{4}{6^0},\frac{10}{6^1},\frac{0}{6^2},\frac{4}{6^3},\frac{1}{6^4},\frac{1}{6^5}$$

$$=\frac{4}{6^0},\frac{4}{6^1},\frac{1}{6^2},\frac{4}{6^3},\frac{1}{6^4},\frac{1}{6^5}$$

Question 2: Strategy 3

Extend the stacked algorithm:

$$\begin{array}{ccccccc} & {}^{1} & 3^{1} & 5 & 0^{1} & 5 & 2 \\ * & & & & & & 2 \\ \hline & 1 & 1 & 4 & 1 & 4 & 4 \end{array}$$

Question 3

(3) LBCC has a standing order with the *Cola Corporation* that whenever an order of size S is shipped to CSULB, an order of size $S/3$ is shipped to LBCC. How will the order be packed for shipping?

Question 3: Strategy 1

In base-6, $\frac{1}{3} * x = (\frac{1}{6} * x) * 2$:

$$\frac{1}{3}\left(\frac{1}{6^5}, \frac{1}{6^4}, \frac{4}{6^3}, \frac{1}{6^2}, \frac{4}{6^1}, \frac{4}{6^0}\right) = \frac{2}{6^4}, \frac{3}{6^3}, \frac{2}{6^2}, \frac{3}{6^1}, \frac{3+\frac{1}{3}}{6^0}$$

$$= \frac{2}{6^4}, \frac{3}{6^3}, \frac{2}{6^2}, \frac{3}{6^1}, \frac{3}{6^0}. \frac{2}{6^{-1}}$$

Question 3: Strategy 2

Extending long division:

$$
\begin{array}{l}
\ 2\\
3\overline{)114144_6}\\
10
\end{array},
\quad
\begin{array}{l}
\ 2\\
3\overline{)114144_6}\\
\underline{10}\\
14144_6
\end{array},
\quad
\begin{array}{l}
\ 23\\
3\overline{)114144_6}\\
\underline{10}\\
14144_6\\
\underline{13}\\
1144_6
\end{array},
\quad \ldots, \quad
\begin{array}{l}
\ 23233_6\\
3\overline{)114144_6}\\
\underline{10}\\
14144_6\\
\underline{13}\\
1144_6\\
\underline{10}\\
144_6\\
\underline{13}\\
14_6\\
\underline{13}\\
1_6
\end{array}
$$

Question 3: Strategy 2

Extending long division:

$$
\begin{array}{r}
23233.2_6 \\
3\overline{)114144.0_6} \\
\underline{10} \\
14144_6 \\
\underline{13} \\
1144_6 \\
\underline{10} \\
144_6 \\
\underline{13} \\
14_6 \\
\underline{13} \\
1.0_6 \\
\underline{1.0} \\
.0
\end{array}
$$

Computational Technology

WolframAlpha computational intelligence.

convert (10000/3) to base 6

NATURAL LANGUAGE | MATH INPUT | EXTENDED KEYBOARD | EXAMPLES | UPLOAD | RANDOM

Input interpretation

convert $\frac{10000}{3}$ to base 6

Result

23233.2_6

Other base conversions — Show digit key | More digits | More bases

$1.1010000010101\ldots_2 \times 2^{11}$

$310011.11111111\ldots_4$

$6405.2525252525\ldots_8$

$1b19.4_{12}$

$d05.55555555555\ldots_{16}$

Extensions!

What does this make you wonder?

What does this make you want to explore?

M's Frustration/Conjecture

$$0.\overline{3}_{10} = \frac{1}{3} = 0.2_6$$

Question

How can we tell when a fraction will have a finite or infinite representation? More specifically, what properties of n and b guarantee a finite (or infinite) representation of $\frac{1}{n}$ in base-b?

E1: (In)finite dec/heximals

base→	2	3	4	5	6	7
$1/2$	$\mathbf{0.1_2}$	$0.\overline{1}_3$	$\mathbf{0.2_4}$	$0.\overline{2}_5$	$\mathbf{0.3_6}$	$0.\overline{3}_7$
$1/3$	$0.\overline{01}_2$	$\mathbf{0.1_3}$	$0.\overline{1}_4$	$0.\overline{13}_5$	$\mathbf{0.2_6}$	$0.\overline{2}_7$
$1/4$	$\mathbf{0.01_2}$	$0.\overline{02}_3$	$\mathbf{0.1_4}$	$0.\overline{1}_5$	$\mathbf{0.13_6}$	$0.\overline{15}_7$
$1/5$	$0.\overline{0011}_2$	$0.\overline{0121}_3$	$0.\overline{03}_4$	$\mathbf{0.1_5}$	$0.\overline{1}_6$	$0.\overline{1254}_7$
$1/6$	$0.0\overline{01}_2$	$0.0\overline{1}_3$	$0.0\overline{2}_4$	$0.\overline{04}_5$	$\mathbf{0.1_6}$	$0.\overline{1}_7$
$1/7$	$0.\overline{001}_2$	$0.\overline{010212}_3$	$0.\overline{021}_4$	$0.\overline{032412}_5$	$0.\overline{05}_6$	$\mathbf{0.1_7}$

Theorem (M's Conjecture)

For integers n and b greater than 1, the fraction $\frac{1}{n}$ has a finite representation in base-b if an only if n divides a power b. Moreover, the length of the finite representation (number of digits in the representation before the start of a tail of all 0s) is the smallest power ℓ such that $n|b^\ell$.

E2: Patterns in dec/heximals

base→	2	3	4	5	6	7
$1/2$	$\mathbf{0.1_2}$	$\mathbf{0.\overline{1}_3}$	0.2_4	$0.\overline{2}_5$	0.3_6	$0.\overline{3}_7$
$1/3$	$\mathbf{0.\overline{01}_2}$	$\mathbf{0.1_3}$	$\mathbf{0.\overline{1}_4}$	$0.\overline{13}_5$	0.2_6	$0.\overline{2}_7$
$1/4$	0.01_2	$\mathbf{0.\overline{02}_3}$	$\mathbf{0.1_4}$	$\mathbf{0.\overline{1}_5}$	0.13_6	$0.\overline{15}_7$
$1/5$	$0.\overline{0011}_2$	$0.\overline{0121}_3$	$\mathbf{0.\overline{03}_4}$	$\mathbf{0.1_5}$	$\mathbf{0.\overline{1}_6}$	$0.\overline{1254}_7$
$1/6$	$0.0\overline{01}_2$	$0.0\overline{1}_3$	$0.0\overline{2}_4$	$\mathbf{0.\overline{04}_5}$	$\mathbf{0.1_6}$	$\mathbf{0.\overline{1}_7}$
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Theorem

- $\frac{1}{b} = 0.1_b$
- $\frac{1}{b-1} = 0.\overline{1}_b$
- $\frac{1}{b+1} = 0.\overline{0(b-1)}_b$

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1/2	$\mathbf{0.1_2}$	$0.\overline{1}_3$	0.2_4	$0.\overline{2}_5$	0.3_6	$0.\overline{3}_7$
1/3	$0.\overline{01}_2$	$\mathbf{0.1_3}$	$0.\overline{1}_4$	$0.\overline{13}_5$	0.2_6	$0.\overline{2}_7$
1/4	$\mathbf{0.01_2}$	$0.\overline{02}_3$	$\mathbf{0.1_4}$	$0.\overline{1}_5$	0.13_6	$0.\overline{15}_7$
1/5	$0.\overline{0011}_2$	$0.\overline{0121}_3$	$0.\overline{03}_4$	$\mathbf{0.1_5}$	$0.\overline{1}_6$	$0.\overline{1254}_7$
1/6	$0.0\overline{01}_2$	$0.0\overline{1}_3$	$0.0\overline{2}_4$	$0.\overline{04}_5$	$\mathbf{0.1_6}$	$0.\overline{1}_7$
1/7	$0.\overline{001}_2$	$0.\overline{010212}_3$	$0.\overline{021}_4$	$0.\overline{032412}_5$	$0.\overline{05}_6$	$\mathbf{0.1_7}$

Theorem

- $\frac{1}{b} = 0.1_b$; $\frac{1}{b^2} = 0.01_b$
- $\frac{1}{b-1} = 0.\overline{1}_b$
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E2: Patterns in dec/heximals

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1/3	$0.\overline{01}_2$	0.1_3	$\mathbf{0.\overline{1}_4}$	$0.\overline{13}_5$	0.2_6	$0.\overline{2}_7$
1/4	0.01_2	$0.\overline{02}_3$	0.1_4	$\mathbf{0.\overline{1}_5}$	0.13_6	$0.\overline{15}_7$
1/5	$0.\overline{0011}_2$	$0.\overline{0121}_3$	$0.\overline{03}_4$	0.1_5	$\mathbf{0.\overline{1}_6}$	$0.\overline{1254}_7$
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Theorem

- $\frac{1}{b} = 0.1_b$; $\frac{1}{b^2} = 0.01_b$
- $\frac{1}{b-1} = 0.\overline{1}_b$; *What if* $b = 2$*?*
- $\frac{1}{b+1} = 0.\overline{0(b-1)}_b$

ColaCorp

BK

E2: Patterns in dec/heximals

base→	2	3	4	5	6	7
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1/3	$\mathbf{0.\overline{01}_2}$	0.1_3	$0.\overline{1}_4$	$0.\overline{13}_5$	0.2_6	$0.\overline{2}_7$
1/4	0.01_2	$\mathbf{0.\overline{02}_3}$	0.1_4	$0.\overline{1}_5$	0.13_6	$0.\overline{15}_7$
1/5	$0.\overline{0011}_2$	$0.\overline{0121}_3$	$\mathbf{0.\overline{03}_4}$	0.1_5	$0.\overline{1}_6$	$0.\overline{1254}_7$
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1/7	$0.\overline{001}_2$	$0.\overline{010212}_3$	$0.\overline{021}_4$	$0.\overline{032412}_5$	$\mathbf{0.\overline{05}_6}$	0.1_7

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1/5	$0.\overline{0011}_2$	$0.\overline{0121}_3$	$\mathbf{0.\overline{03}_4}$	$\mathbf{0.1_5}$	$\mathbf{0.\overline{1}_6}$	$0.\overline{1254}_7$
1/6	$0.0\overline{01}_2$	$0.0\overline{1}_3$	$0.0\overline{2}_4$	$\mathbf{0.\overline{04}_5}$	$\mathbf{0.1_6}$	$\mathbf{0.\overline{1}_7}$
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Theorem

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- $\frac{1}{b-1} = 0.\overline{1}_b$
- $\frac{1}{b+1} = 0.\overline{0(b-1)}_b$

$$(b+1)(b-1) = b^2 - 1$$

E3: Divisibility Tests

Let $d = d_m d_{m-1} \ldots d_2 d_1 d_0$ be positive integer with base-10 digits d_i. Then d is divisible...

- by 2, 5, or 10 IFF d_0 is divisible by 2, 5, or 10 respectively; by 4 IFF the two-digit number $d_1 d_0$ is divisible by 4; and by 8 IFF the three-digit number $d_2 d_1 d_0$ is divisible by 8.
- by 3 IFF $s = d_m + d_{m-1} + \cdots + d_1 + d_0$ is divisible by 3; and by 9 IFF s is divisible by 9.
- by 11 IFF $a = d_m - d_{m-1} + \cdots \pm d_1 \mp d_0$ is divisible by 11.
- by 7 IFF the new integer $j = d_m d_{m-1} \ldots d_2 d_1 + 5d_0$ is divisible by 7. (Chika's Test)
- by 6 IFF it is divisible by both 2 and 3; and by 12 IFF it is divisible by both 3 and 4.

E3: Student Conjectures

Conjecture

For any base b and any factor f that divides b, an integer d is divisibly by f if and only if the b^0 digit of d's base-b representation is divisible by f.

Conjecture

For any base b and any integer $d = d_m d_{m-1} \ldots d_2 d_1 d_0$ written in base-b notation, d is divisible by $b - 1$ if and only if $s = d_m + d_{m-1} + \cdots + d_2 + d_1 + d_0$ is divisible by $b - 1$.

Conjecture

For any base b and any integer $d = d_m d_{m-1} \ldots d_2 d_1 d_0$ written in base-b notation, d is divisible by $b + 1$ if and only if $a = d_m - d_{m-1} + \cdots \pm d_2 \mp d_1 \pm d_0$ is divisible by $b + 1$.

Why use this activity?

- Conceptual learning (eg place value)
- Community, collaboration, inquiry, ...
- Depth and connectivity
- Decentering/perspective-taking

Thank You!

Thank you!

Questions/Suggestions?
Requests for links or resources?

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References

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- Larnyoh, M. T. (2019, November 11). *Chika Ofili: 12-year-old Nigerian awarded for making mathematics easy to learn*. PulseAfrica. https://www.pulse.com.gh/bi/lifestyle/chika-ofili-12-year-old-nigerian-awarded-for-making-mathematics-easy-to-learn/w246s09